Real Analysis 20-11-06.

§ 3.4 Hausdorff measures.

Def (Hausdorff, 1918)

Let $A \subset \mathbb{R}^n$, $\delta > 0$, $s \in [0, \infty)$, define

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} |A_i|^s : A \subset \bigcup_{i=1}^{\infty} A_i, \ |A_i| < \delta \right\}.$$

(here $|A_i| := \mathrm{diam}(A_i)$)

and

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

(Using the fact $\mathcal{H}^s_\delta(A) \uparrow \mathcal{H}^s(A)$ as $\delta \downarrow 0$)

We call $\mathcal{H}^s(A)$ the s-dimensional Hausdorff measure of A.

Thm 3.7. (a) $\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$ for each $s \geq 0$.

(b) Suppose $\mathcal{H}^s(A) < \infty$, then $\exists$ a Borel set B such that $B \supset A$ and

$$\mathcal{H}^s(B) = \mathcal{H}^s(A).$$

(c) For any open set $G \subset \mathbb{R}^n$,

$$\mathcal{H}^s(G) = \sup\{\mathcal{H}^s(K) : K \text{ compact}, K \subset G\}.$$

(d) If A is a Borel set with $\mathcal{H}^s(A) < \infty$, then $\forall\, \varepsilon > 0$, $\exists$ compact $K \subset A$ such that

$$\mathcal{H}^s(A \setminus K) < \varepsilon.$$

Pf. (a). First we show that $\mathcal{H}^s_\delta$ is an outer measure.

This is clear since $\mathcal{H}^s_\delta$ is generated by a gauge $(\mathcal{R}_\delta, |\cdot|^s)$, where

$$\mathcal{R}_\delta = \{A \subset \mathbb{R}^n : \operatorname{diam}(A) < \delta\}.$$

We claim that $\mathcal{H}^s$ is also an outer measure.

Clearly, $\mathcal{H}^s(\emptyset) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(\emptyset) = 0.$

Next for $A \subset \bigcup_{i=1}^\infty A_i$, then

$$\mathcal{H}^s_\delta(A) \leq \sum_{i=1}^\infty \mathcal{H}^s_\delta(A_i)$$

$$\leq \sum_{i=1}^\infty \mathcal{H}^s(A_i).$$

Letting $\delta \to 0$ gives $\mathcal{H}^s(A) \leq \sum_{i=1}^\infty \mathcal{H}^s(A_i)$.

Hence $\mathcal{H}^s$ is an outer measure.

Next we show that $\mathcal{H}^s$ is a metric outer measure.

To see this, let $A, B \subset \mathbb{R}^n$ with $d(A,B) > 0$.

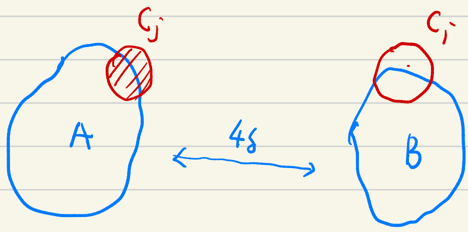
Take $0 < \delta < \dfrac{d(A,B)}{4}$.

Suppose $A \cup B \subset \bigcup_{k=1}^{\infty} C_k$ with $|C_k| < \delta$.

Write $\mathcal{A} = \{ C_j : C_j \cap A \neq \emptyset \}$

$\mathcal{B} = \{ C_j : C_j \cap B \neq \emptyset \}$.

Since $|C_j| < \delta$, $d(A,B) > 4\delta$, so no C_j intersects both A and B.



Hence $\displaystyle\sum_{k=1}^{\infty} |C_k|^s \geq \sum_{C_j \in \mathcal{A}} |C_j|^s + \sum_{C_j \in \mathcal{B}} |C_j|^s$

Notice that $\displaystyle\bigcup_{C_j \in \mathcal{A}} C_j \supset A$, $\displaystyle\bigcup_{C_j \in \mathcal{B}} C_j \supset B$.

So $\displaystyle\sum_{k=1}^{\infty} |C_k|^s \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$

Taking infimum over the covers $\{C_k\}$ of $A\cup B$ gives

$$\mathcal{H}^s_\delta(A\cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B),$$

so

$$\mathcal{H}^s_\delta(A\cup B) = \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).$$

Letting $\delta\to 0$ gives

$$\mathcal{H}^s(A\cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Hence $\mathcal{H}^s$ is a metric outer measure. So it is a Borel measure.

This proves (a).

Now we prove (b): If $\mathcal{H}^s(A)<\infty$, then $\exists$ a Borel $B\supset A$ with $\mathcal{H}^s(B) = \mathcal{H}^s(A)$.

Notice that for $\delta>0$, $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s(A) < \infty$.

Moreover for $C\subset \mathbb{R}^n$, $|C| = |\bar{C}|$, where $\bar{C}$ denotes the closure of C.

Hence for any integer $k>0$, by definition, we can find $\{C_j^k\}_{j=1}^\infty$ such that C_j^k are closed sets,

$|C_j^k| < \frac{1}{k}$, and $A \subset \bigcup_{j=1}^{\infty} C_j^k$, moreover

$$\sum_{j=1}^{\infty} |C_j^k|^s \leq \mathcal{H}^s_{1/k}(A) + \frac{1}{k} \leq \mathcal{H}^s(A) + \frac{1}{k}$$

Define $B_k = \bigcup_{j=1}^{\infty} C_j^k$, then B_k is Borel,

and $A \subset B_k$.

Let $B = \bigcap_{k=1}^{\infty} B_k$, then B is Borel, $B \supset A$.

Notice that for each $k \in \mathbb{N}$,

$$\begin{aligned} \mathcal{H}^s_{1/k}(B) &\leq \mathcal{H}^s_{1/k}(B_k) \\ &\leq \sum_{j=1}^{\infty} |C_j^k|^s \\ &\leq \mathcal{H}^s(A) + 1/k \end{aligned}$$

Letting $k \to \infty$ gives

$$\mathcal{H}^s(B) \leq \mathcal{H}^s(A),$$

and so $\mathcal{H}^s(B) = \mathcal{H}^s(A)$. This proves (b).

(c) For open $G \subset \mathbb{R}^n$,

(*) $\mathcal{H}^s(G) = \sup\{ \mathcal{H}^s(K) : K \text{ compact}, K \subset G \}$.

To prove (*), it suffices to show that

$\exists$ a sequence of compact sets (K_j) such that

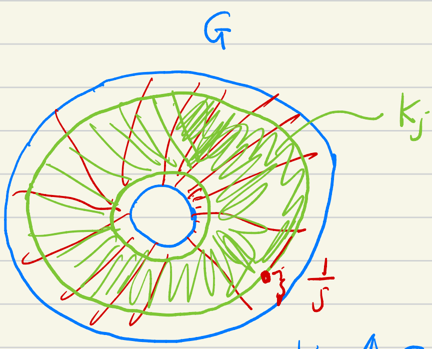
$$K_j \uparrow G$$

$\left(\text{i.e. } K_{j+1} \supset K_j \text{ and } G = \bigcup_{j=1}^{\infty} K_j \right)$

Then $\mathcal{H}^s(G) = \lim_{j\to\infty} \mathcal{H}^s(K_j)$ by the cty of measure

Now we construct such K_j as follows:

$$K_j = \{ x \in \mathbb{R}^n : d(x, G^c) \geq \tfrac{1}{j}, |x| \leq j \}.$$



A direct further check shows that $K_j \uparrow G$.

(d) If $\mathcal{H}^s(A)<\infty$, A is Borel, then $\forall\ \varepsilon>0$, $\exists$ compact $K\subset A$ so that

$$\mathcal{H}^s(A\setminus K)<\varepsilon.$$

Actually this is a general property for all Borel measures on $\mathbb{R}^n$. You are referred to [Evans - Gariepy] Lem 1.1 (i), P. 6.

Prop 3.8. Let $A\subset\mathbb{R}^n$. Then

(1) $\mathcal{H}^s(TA)=\mathcal{H}^s(A)$ if T is a Euclidean motition (i.e. $Tx=Ux+b$, where U is an orthogonal transformation).

(2) $\mathcal{H}^s(\lambda A)=\lambda^s\mathcal{H}^s(A)$, $\forall\ \lambda>0$.

Prop 3.9. Let $A \subset \mathbb{R}^n$. Then

(1) $\mathcal{H}^s(A) = 0$, if $s > n$

(2) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$ if $t > s$.

(3) If $\mathcal{H}^s(A) > 0$, then $\mathcal{H}^t(A) = \infty$ if $t < s$.

Let $s > n$.

Pf. (1) We prove that $\mathcal{H}^s(\mathbb{R}^n) = 0$.

Notice that $\mathbb{R}^n$ is the countable union.

$$\bigcup_{z \in \mathbb{Z}^n} \left([0,1]^n + z \right)$$

It is enough to show that

$$\mathcal{H}^s\left([0,1]^n \right) = 0. \qquad (**)$$

Notice that for $k \in \mathbb{N}$, $[0,1]^n$ can be covered by k^n many subcubes of side $1/k$

Each such subcube is of diameter $\frac{\sqrt{n}}{k}$.

Hence

$$\mathcal{H}^s_{\sqrt{n}/k}([0,1]^n) \leq k^n \cdot \left(\frac{\sqrt{n}}{k}\right)^s$$

$$= (\sqrt{n})^s \cdot k^{n-s}$$

$$\to 0 \quad \text{as} \quad k \to \infty$$

It follows that $\mathcal{H}^s([0,1]^n) = 0$.

(b) If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$ if $t > s$.

Assume $t > s$. Let $\delta > 0$. Then we can find a δ-cover $\{C_i\}_{i=1}^{\infty}$ of A such that

$$\sum_{i=1}^{\infty} |C_i|^s \leq \mathcal{H}^s_\delta(A) + 1.$$

Then $$\sum_{i=1}^{\infty} |C_i|^t = \sum_{i=1}^{\infty} |C_i|^s \cdot |C_i|^{t-s}$$

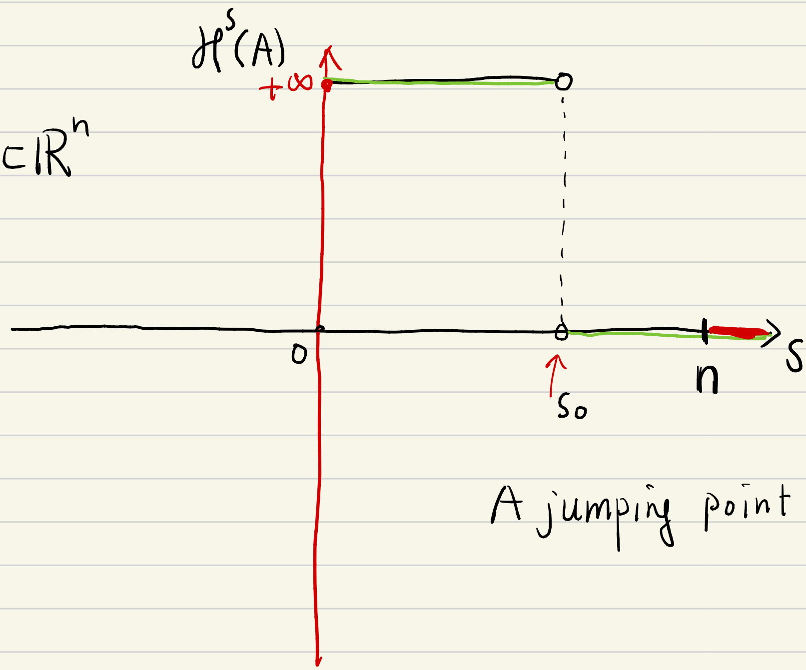
$$\leq \delta^{t-s} \cdot \sum_{i=1}^{\infty} |C_i|^s$$

$$\leq \left(\mathcal{H}^s_\delta(A) + 1\right) \cdot \delta^{t-s}$$

So
$$\mathcal{H}^t_\delta(A) \le \sum_{i=1}^{\infty} |C_i|^t \le \left(\mathcal{H}^s_\delta(A)+1\right)\cdot \delta^{t-s}$$
$$\le \left(\mathcal{H}^s(A)+1\right) \delta^{t-s}.$$

Letting $\delta \to 0$ gives
$$\mathcal{H}^t(A) = 0.$$

$A \subset \mathbb{R}^n$



A jumping point

Figure. 1

Prop 3.10. (a) $\mathcal{H}^0$ is the counting measure on $\mathbb{R}^n$.

(b) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$.

(c) $\mathcal{H}^n = C(n) \cdot \mathcal{L}^n$ on $\mathbb{R}^n$, where $C(n)$ is a positive constant.

Pf. (a) follows from the definition.

(b) follows from the fact that

if $A \subset \mathbb{R}$, and $\{C_i\}$ is a cover of A,

then $\{[a_i, b_i]\}$ is also a cover of A

where $a_i = \inf C_i$, $b_i = \sup C_i$

and $\sum |C_i|^1 = \sum_i |b_i - a_i|$.

This property implies that $\mathcal{H}^1 = \mathcal{L}^1$, using the fact

$$\mathcal{L}^1(A) = \inf\left\{ \sum_i |b_i - a_i| : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i], \ b_i - a_i < \delta \right\}$$

$\forall \delta > 0$.

(3) Since $\mathcal{H}^n$ is a translation invariant Borel measure, so $\exists$ $C(n)$ such that

$$\mathcal{H}^n = C(n)\, \mathcal{L}^n.$$

To see that $C(n)$ is a positive number, it is enough to show that

$$0 < \mathcal{H}^n([0,1]^n) < \infty.$$

By dividing $[0,1]^n$ into k^n many subcubes of side $\frac{1}{k}$ gives

$$\mathcal{H}^n_{\sqrt{n}/k}([0,1]^n) \leq k^n \cdot \left(\frac{\sqrt{n}}{k}\right)^n$$

$$\leq (\sqrt{n})^n < \infty$$

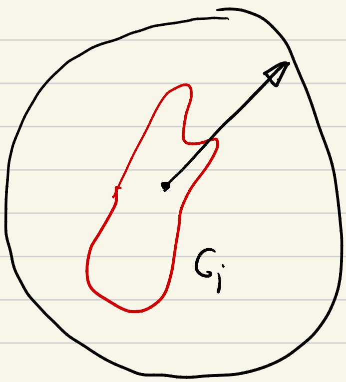
Letting $k \to \infty$ gives

$$\mathcal{H}^n([0,1]^n) \leq (\sqrt{n})^n.$$

Next we estimate ~~the~~ lower bound of $\mathcal{H}^n([0,1]^n)$.

Let $\{C_i\}$ be a δ-cover of $[0,1]^n$.

For each i, let B_i be a ball of radius $\operatorname{diam} C_i$ and so that $B_i \supset C_i$



Then

$$\sum_i |C_i|^n = 2^{-n} \sum_i |B_i|^n$$

$$= d_n 2^{-n} \cdot \sum_i \mathcal{L}^n(B_i)$$

$$\geq d_n 2^{-n} \cdot \mathcal{L}^n([0,1]^n)$$

$$= d_n 2^{-n}.$$